# CONFIGURATION SPACES OF DIVISORS AND HOLOMORPHIC MAPS(Recent development of algebraic topology)

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# CONFIGURATION SPACES OF DIVISORS AND HOLOMORPHIC MAPS

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## §1. Introduction.

Spaces of holomorphic maps between complex manifolds have played a major role in such diverse branches of mathematics as analysis, differential geometry, topology and mathematical physics.

If $X \subset \mathbb{C}P^n$ is a projective variety we denote by $\mathrm{Hol}_d^*(\mathrm{S}^2, X)$ ($\mathrm{Hol}_d(\mathrm{S}^2, X)$) the space of all based (non-based) holomorphic maps $\mathrm{S}^2 \to X$ of degree $d$, where $\mathrm{S}^2$ is the Riemann sphere, $\mathrm{S}^2 = \mathbb{C} \cup \{\infty\}$. For simplicity we shall assume that the degree $d$ is a non-negative integer. The corresponding space of based (non-based) continuous maps of degree $d$ is denoted by $\mathrm{Map}_d^*(\mathrm{S}^2, X)$ ($\mathrm{Map}_d(\mathrm{S}^2, X)$).

In [S] Segal studied the inclusion map

$$I_d : \mathrm{Hol}_d^*(\mathrm{S}^2, \mathbb{C}P^n) \longrightarrow \mathrm{Map}_d^*(\mathrm{S}^2, \mathbb{C}P^n)$$

and showed that this inclusion map is a homotopy equivalence up to dimension $d(2n-1)$.

For any projective variety $X$ it is natural to consider the following

**Problem.** *When is the map $I_d : \mathrm{Hol}_d^*(\mathrm{S}^2, X) \to \mathrm{Map}_d^*(\mathrm{S}^2, X)$ a homotopy equivalence up to some dimension $n(d)$, such that $n(d) \to \infty$ as $d \to \infty$?*

Segal's results for $X = \mathbb{C}P^n$ have been generalized to the case when $X$ is a Grassmanian, or more generally, a flag manifold (see [G], [Gu], [K], [M²]). In this note we consider the case where $X$ is the complement of a union of linear subspaces in $\mathbb{C}P^n$: $X = \mathbb{C}P^n \setminus \bigcup_{\alpha \in \Lambda} H_\alpha$, where $\{H_\alpha : \alpha \in \Lambda\}$ is a family of linear subspaces of $\mathbb{C}P^n$. Our purpose is to announce the main results of [GKY], which extend Segal's results [S] to the case $X = X_n = \mathbb{C}P^n \setminus \bigcup_{0 \le i < j \le n} H_{i,j}$, where $H_{i,j} = \{[z_0 : z_1 : \cdots : z_n] \in \mathbb{C}P^n : z_i = z_j = 0\}$.

The precise statements of our results are as follows:

**Theorem 1.** *The inclusion maps*

$$I_d : \mathrm{Hol}_d^*(S^2, X_n) \longrightarrow \mathrm{Map}_d^*(S^2, X_n)$$

*and*

$$J_d : \mathrm{Hol}_d(S^2, X_n) \longrightarrow \mathrm{Map}_d(S^2, X_n)$$

*are homology equivalences up to dimension d.*

**Theorem 2.** *If $2d > n$ the two maps above are homotopy equivalences up to dimension d.*

Here we call an inclusion map $X \longrightarrow Y$ a *homotopy equivalence* (*homology equivalence*) *up to dimension* $m$ if $\pi_j(Y, X) = 0$ when $j \leq m$ (if $H_j(Y, X) = 0$ when $j \leq m$).

**Remark.**

(1) *For $n = 1$ the above results were obtained in* [S].
(2) *We expect that similar methods can be used to obtain analogous results when $X = \mathbb{C}P^n \setminus \cup_I P(I)$, where $P(I) = \{[z_0 : \cdots : z_n] \in \mathbb{C}P^n : p_j = 0 \text{ if } j \in I\}$, and the union is over a collection of subsets $I$ of $\{0, 1, 2, \ldots, n\}$.*

## §2. Configuration Spaces of Divisors.

**Definition 2.1.** *For a connected pair of CW-complexes $(X, Y)$, let $Sp^d(X, Y)$ denote the d-fold symmetric product of $X/Y$. Adding a base point gives rise to a natural inclusion $Sp^d(X, Y) \to Sp^{d+1}(X, Y)$ and we put $Sp^\infty(X, Y) = \cup_{d \geq 1} Sp^d(X, Y)$. We define a space $Q_d^{(n)}(X, Y)$ by*

$$Q_d^{(n)}(X, Y) = \{(\xi_0, \ldots, \xi_n) \in \left(Sp^d(X, Y)\right)^{n+1} : \xi_i \cap \xi_j = \emptyset \text{ if } i \neq j\}.$$

*If $Y = \emptyset$, we write $Sp^d(X) = Sp^d(X, \emptyset)$ and $Q_d^{(n)}(X) = Q_d^{(n)}(X, \emptyset)$.*

If $M$ is a connected open manifold, adding $(n+1)$ distinct points "from infinity" (c. f. [Mc]) gives a natural stabilization map $i_d : Q_d^{(n)}(M) \to Q_{d+1}^{(n)}(M)$ and we define $\hat{Q}^{(n)}(M)$ to be the "identity component" of $\lim\limits_{d \to \infty} Q_d^{(n)}(M)$. Let $F(X, m)$ be the configuration space of $m$-tuples of distinct points in $X$. In particular, $Q_1^{(n)}(X) = F(X, n+1)$, and it is well-known that $\pi_1\big(F(\mathbb{C}, m)\big) = I(m)$, where $I(m)$ denotes the group of pure braids of $m$ strings. Then we have

**Proposition 2.2.**

(1) $\mathrm{Hol}_d^*(S^2, X_n) = Q_d^{(n)}(\mathbb{C})$.
(2) $\pi_1\big(\mathrm{Hol}_d^*(S^2, X_n)\big) = \begin{cases} I(n+1) & \text{if } d = 1 \\ \mathbb{Z}^{n(n+1)/2} & \text{if } d \geq 2. \end{cases}$

(Part (2) is proved in [E].)

## §3 The Stabilization Theorem.

**Theorem 3.1.** ([GKY],[Ko]). *The stabilization map $i_d : \mathrm{Hol}_d^*(S^2, X_n) \to \mathrm{Hol}_{d+1}^*(S^2, X_n)$ is a homology equivalence up to dimension $d$.*

Using the McDuff-Segal transfer ([Mc],[S]) we obtain

**Proposition 3.2.** *For any commutative ring $R$, the induced homomorphism $i_{d*} : H_*(\mathrm{Hol}_d^*(S^2, X_n), R) \to H_*(\mathrm{Hol}_{d+1}^*(S^2, X_n), R)$ is a split monomorphism. More precisely, there is a family of graded $R$-modules $\{R_m : m \geq 0\}$ such that*

(a) $H_*(\mathrm{Hol}_d^*(S^2, X_n), R) = \bigoplus_{0 \leq m \leq d} R_m$.

(b) *The above isomorphism is compatible with the splitting monomorphism.*

These results lead us to expect

**Conjecture 3.3.** *There is a stable splitting*

$$\mathrm{Hol}_d^*(S^2, X_n) \underset{S}{\simeq} \bigvee_{1 \leq j \leq d} D_j(n)$$

*such that*

$$D_j(n) \underset{S}{\simeq} \mathrm{Hol}_j^*(S^2, X_n)/\mathrm{Hol}_{j-1}^*(S^2, X_n).$$

**Remark 3.4.** ([$C^2M^2$]) *This is true for $n = 1$ .*

## §4. The Scanning Map.

**Definition 4.1.** *Let $\varepsilon > 0$ be any positive real number, and let $D_z(\varepsilon)$ denote the open disk of radius $\varepsilon$ with centre at $z \in \mathbb{C}$. Define the map $S_d : Q_d^{(n)}(\mathbb{C}) \times \mathbb{C} \to Q^{(n)}(S^2, \infty)$ by*

$$((\xi_0, \dots, \xi_n), z) \mapsto (\xi_0 \cap D_z(\varepsilon), \dots, \xi_n \cap D_z(\varepsilon)) \in Q^{(n)}(\bar{D}_z(\varepsilon), \partial \bar{D}_z(\varepsilon)) \simeq Q^{(n)}(S^2, \infty).$$

*Since $\lim_{z \to \infty} S_d(\Xi, z) = (\emptyset, \emptyset, \dots, \emptyset)$ for any $\Xi \in Q_d^{(n)}(\mathbb{C})$, we define $S_d(\Xi, \infty) = (\emptyset, \emptyset, \dots, \emptyset)$ and obtain a map*

$$S_d : Q_d^{(n)}(\mathbb{C}) \times S^2 \to Q^{(n)}(S^2, \infty).$$

*Taking the adjoint we obtain a map*

$$S_d : Q_d^{(n)}(\mathbb{C}) \to \mathrm{Map}_d^*(S^2, Q^{(n)}(S^2, \infty)).$$

*Its homotopy class is independent of the choice of $\varepsilon$. We call $S_d$ the scanning map.*

It can be shown that $Q^{(n)}(S^2, \infty) \simeq \overset{n+1}{\vee} \mathbb{C}P^\infty$. It is also easy to see that there is a homotopy equivalence $\alpha_d : \Omega^2_d(\overset{n+1}{\vee} \mathbb{C}P^n) \simeq \Omega^2_{d+1}(\overset{n+1}{\vee} \mathbb{C}P^n)$ such that the following diagram is commutative up to homotopy

$$\begin{array}{ccc} Q^{(n)}_d(\mathbb{C}) & \xrightarrow{S_d} & \Omega^2_d(\overset{n+1}{\vee} \mathbb{C}P^\infty) \\ \big\downarrow{\scriptstyle i_d} & & \simeq \big\downarrow{\scriptstyle \alpha_d} \\ Q^{(n)}_{d+1}(\mathbb{C}) & \xrightarrow{S_{d+1}} & \Omega^2_{d+1}(\overset{n+1}{\vee} \mathbb{C}P^\infty) \end{array}$$

Consider the mapping telescope of the maps

$$Q^{(n)}_1(\mathbb{C}) \xrightarrow{i_1} Q^{(n)}_2(\mathbb{C}) \xrightarrow{i_2} Q^{(n)}_3(\mathbb{C}) \xrightarrow{i_3} Q^{(n)}_4(\mathbb{C}) \to \dots$$

It is easy to see that this mapping telescope is homotopy equivalent to $\hat{Q}^{(n)}$. Hence we obtain a stabilized scanning map

$$\hat{S} : \hat{Q}^{(n)} \to \Omega^2_0(\overset{n+1}{\vee} \mathbb{C}P^\infty).$$

By arguing exactly as in [S], we obtain

**Proposition 4.3.** *The scanning map $\hat{S}$ is a homotopy equivalence.*

*Sketch proofs of Theorems 1 and 2.* Let $G = (\mathbb{C}^*)^n$ and define a $G$-action on $X_n$ by

$$((t_1, \dots, t_n), [p_0 : \dots : p_n]) \mapsto [p_0 : t_1 p_1 : \dots : t_n p_n].$$

Then there is a fibre sequence

$$T^n \to X_n \xrightarrow{q} \overset{n+1}{\vee} \mathbb{C}P^\infty.$$

(This follows from the fact that $EG \times_G X_n \simeq \overset{n+1}{\vee} \mathbb{C}P^\infty$). There is a homotopy commutative diagram:

$$\begin{array}{ccc} \mathrm{Hol}^*_d(S^2, X_n) & \xrightarrow{I_d} & \mathrm{Map}^*_d(S^2, X_n) = \Omega^2_d X_n \\ \simeq \big\downarrow & & \simeq \big\downarrow{\scriptstyle \Omega^2 q} \\ Q^{(n)}_d(\mathbb{C}) & \xrightarrow{S_d} & \Omega^2_d(\vee^{n+1} \mathbb{C}P^\infty) \end{array}$$

It follows that $\lim\limits_{d \to \infty} I_d$ is a homotopy equivalence. Hence Theorem 1 follows from the stabilization theorem.

Finally, an argument analogous to the one given by Segal in [S] shows that the space $Q^{(n)}_d(\mathbb{C})$ is nilpotent up to dimension $d$ if $2d > n$. Theorem 2 follows from the Whitehead Theorem [HR] $\square$

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